ON THE COMPARISON IN MEAN RESIDUAL LIFE ORDER OF THE CONVOLUTIONS OF GEOMETRIC RANDOM VARIABLES

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Abstract

Let \(X_1, \ldots, X_n\) and \(Y_1, \ldots, Y_n\) be independent heterogeneous geometric random variables with parameters \(p_1 \leq \cdots \leq p_n\) and \(q_1 \leq \cdots \leq q_n\), respectively. We prove that \(\sum_{i=1}^k X_i\) is larger than \(\sum_{i=1}^k Y_i\), for \(k = 1, \ldots, n\), in mean residual life order if, and only if, the harmonic mean of \(p_1, \ldots, p_k\) is smaller than the harmonic mean of \(q_1, \ldots, q_k\), for \(k = 1, \ldots, n\). In this way, we slightly improve a result of Mao, Hu and Zhao (2010). On the other hand, we present a new proof of this result. For a complete argumentation, we prove a discrete version of a well-known closure property.

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1 Introduction

In the last years, the stochastic comparison of convolutions of some families of parametric distributions has been studied by many researchers. In this vein, necessary or sufficient conditions were expressed for the comparison of these convolutions in terms of some adequate weak majorization type orders. Schur-majorization properties of convolutions of geometric random variables were discussed in [1] and, recently, in [8] and [9]. A comprehensive treatment of majorization and weak majorization orders and their applications has been provided by book [4]. In this direction, Bon and Păltănea introduced in [2] \(p\)-larger order and Zhao and Balakrishnan defined in [7] reciprocal majorization order. The following definition introduces these weak majorization orders and related notions.

Definition 1. Given a vector \(v = (v_1, \ldots, v_n) \in \mathbb{R}^n\), let \(v_{(1)} \leq \cdots \leq v_{(n)}\) denote the increasing arrangement of its components. Then:

1. \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\) is said to majorize \(y = (y_1, \ldots, y_n) \in \mathbb{R}^n\) (denoted by \(x \succeq y\)) if

\[
\sum_{i=1}^k x_{(i)} \leq \sum_{i=1}^k y_{(i)} \quad \text{for } k = 1, \ldots, n - 1, \quad \text{and} \quad \sum_{i=1}^n x_{(i)} = \sum_{i=1}^n y_{(i)};
\]

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(2) $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ is said to majorize weakly $\mathbf{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$ (denoted by $\mathbf{x} \preceq w \mathbf{y}$) if
\[
\sum_{i=1}^{k} x(i) \leq \sum_{i=1}^{k} y(i) \text{ for } k = 1, \cdots, n;
\]

(3) $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}_+^n$ is said to be $p$-larger than $\mathbf{y} = (y_1, \cdots, y_n) \in \mathbb{R}_+^n$ (denoted by $\mathbf{x} \preceq p \mathbf{y}$) if
\[
\prod_{i=1}^{k} x(i) \leq \prod_{i=1}^{k} y(i) \text{ for } k = 1, \cdots, n;
\]

(4) $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}_+^n$ is said to majorize reciprocal $\mathbf{y} = (y_1, \cdots, y_n) \in \mathbb{R}_+^n$ (denoted by $\mathbf{x} \succeq r \mathbf{y}$) if
\[
\sum_{i=1}^{k} \frac{1}{x(i)} \geq \sum_{i=1}^{k} \frac{1}{y(i)} \text{ for } k = 1, \cdots, n;
\]

(5) $f : I^n \to \mathbb{R}$, where $I$ denotes a real interval, is said to be a Schur-convex function if $f(\mathbf{x}) \geq f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in I^n$ for which $\mathbf{x} \succeq w \mathbf{y}$.

The importance of the reciprocal majorization order for the comparison in the mean residual life (MRL) order has been recently pointed out in paper [7], for exponential convolutions, and in paper [3], for geometric convolutions.

In this paper, we present a new proof of the characterization expressed in [3] and we slightly improve this characterization. Thus, we formulate a nice equivalence.

2 Mean residual life ordering of discrete distributions

Let us first recall some definitions for discrete random variables that take on values in $\mathbb{N} = \{0, 1, \cdots\}$. Let $Z$ be such a random variable with a finite mean $\mu_Z$. Let us denote its probability mass function by $f_Z$ and its survival function by $F_Z$. That is $F_Z(n) = P(Z \geq n) = \sum_{i=n}^{\infty} f_Z(i)$ for $n \in \mathbb{N}$. Conventionally, $F_Z(n) = 1$ and $f_Z(n) = 0$ for all $n \in \mathbb{Z} \setminus \mathbb{N}$. The corresponding hazard rate function is defined as
\[
r_Z(n) = \frac{f_Z(n)}{F_Z(n)}, \quad n \in \mathbb{N}.
\]

$Z$ is said to have an Increasing Failure Rate (IFR) if its hazard rate $r_Z(n)$ is increasing in $n \in \mathbb{N}$. Alternatively, $Z$ is IFR if and only if
\[
f_Z(n_2)F_Z(n_1 + k) \geq f_Z(n_1)F_Z(n_2 + k), \quad (1)
\]
for all $n_1, n_2, k \in \mathbb{N}$, such that $n_1 \leq n_2$. Conventionally, $n_1$ and $n_2$ can be assumed in $\mathbb{Z}$.

Suppose that $F_Z(n) > 0$, $\forall \ n \in \mathbb{N}$. The Mean Residual Life (MRL) function of $Z$ is defined as
\[
m_Z(n) = E[Z - n|Z \geq n], \quad n \in \mathbb{N}.
\]
Note that $m_Z(0) = \mu_Z$. In the sequel, we need the following elementary lemma (the proof is omitted).

**Lemma 1.** Let $(a_n)_{n \in \mathbb{Z}}$ and $(b_n)_{n \in \mathbb{Z}}$ be two sequences of positive numbers. For $m \in \mathbb{Z}$, if

$$\sum_{n=m}^{\infty} a_n$$

is convergent and $(b_n)_{n \geq m}$ is bounded then the series $\sum_{n=m}^{\infty} \left[ (b_{n+1} - b_n) \sum_{i=n+1}^{\infty} a_i \right]$ converges, and

$$\sum_{n=m}^{\infty} a_n b_n = b_m \sum_{i=m}^{\infty} a_i + \sum_{n=m}^{\infty} \left[ (b_{n+1} - b_n) \sum_{i=n+1}^{\infty} a_i \right].\quad (2)$$

From Lemma 1 it results

$$\mu_X = \sum_{n=1}^{\infty} F_X(n) < \infty$$

and

$$m_X(n) = \frac{\sum_{i=n+1}^{\infty} F_X(i)}{F_X(n)}, \text{ for } n \in \mathbb{N}.$$}

Let $X$ and $Y$ be two discrete random variables with values in $\mathbb{N}$. Assume that $X$ and $Y$ have finite means and strict positive survival functions. Then $X$ is said to be larger than $Y$ in the *mean residual life order* (denoted by $X \geq_{\text{mrl}} Y$) if

$$m_X(n) \geq m_Y(n), \text{ for all } n \in \mathbb{N},$$

that is

$$\frac{\sum_{i=n+1}^{\infty} F_X(i)}{F_X(n)} \geq \frac{\sum_{i=n+1}^{\infty} F_Y(i)}{F_Y(n)}, \quad \forall \ n \in \mathbb{N}.\quad (3)$$

Generally, the MRL ordering of discrete distributions has corresponding properties to those of the MRL ordering of continuous distributions. Thus, the following equivalences hold (see [6], p. 83)

$$X \geq_{\text{mrl}} Y \iff \frac{\sum_{i=n}^{\infty} F_X(i)}{\sum_{i=n}^{\infty} F_Y(i)} \text{ increases in } n \iff \frac{\sum_{i=t+1}^{\infty} F_X(i)}{F_X(s)} \geq \frac{\sum_{i=t+1}^{\infty} F_Y(i)}{F_Y(s)} \quad \forall \ s \leq t.\quad (4)$$

Note that, if we consider the survival functions as defined on $\mathbb{Z}$, then the above relations are valid for $n, s, t \in \mathbb{Z}$.

Now, we point out a closure property of MRL order for discrete random variables. Firstly, we prove a discrete analog of Lemma 2.A.8 in [6].

**Lemma 2.** Let $X$, $Y$ and $Z$ be three discrete random variables on $\mathbb{N}$, with finite means and strict positive survival functions. Assume that $Z$ is independent of $X$ and $Y$. If $X \geq_{\text{mrl}} Y$ and $Z$ is IFR, then $X + Z \geq_{\text{mrl}} Y + Z$.

**Proof.** From (4), it suffices to show

$$\sum_{i=t}^{\infty} F_{X+Z(i)} \sum_{j=s}^{\infty} F_{Y+Z(j)} \geq \sum_{i=s}^{\infty} F_{X+Z(i)} \sum_{j=t}^{\infty} F_{Y+Z(j)}, \text{ for all } s, t \in \mathbb{N}, \ s \leq t.$$
Conventionally, let us consider the extensions on $\mathbb{Z}$ of the survival functions and distribution functions of $X, Y$ and $Z$. We have

$$
\sum_{i=t}^{\infty} F_{X+Z}(i) = \sum_{i=t}^{\infty} \sum_{n \in \mathbb{Z}} F_X(n) f_Z(i-n) = \sum_{n \in \mathbb{Z}} F_X(n) \sum_{i=t}^{\infty} f_Z(i-n) = \sum_{n \in \mathbb{Z}} F_X(n) F_Z(t-n).
$$

Thus, for $s, t \in \mathbb{N}$ such that $s \leq t$,

$$
\sum_{i=t}^{\infty} F_{X+Z}(i) \sum_{j=s}^{\infty} F_{Y+Z}(j) - \sum_{i=s}^{\infty} F_{X+Z}(i) \sum_{j=t}^{\infty} F_{Y+Z}(j)
$$

$$
= \sum_{m,n \in \mathbb{Z}} F_X(n) F_Y(m) \left[ F_Z(t-n) F_Z(s-m) - F_Z(s-n) F_Z(t-m) \right]
$$

$$
= \sum_{m \in \mathbb{Z}} \sum_{n=m+1}^{\infty} a_n(m) b_n(m),
$$

where

$$
\begin{cases}
    a_n(m) = F_X(n) F_Y(m) - F_X(m) F_Y(n) \\
    b_n(m) = F_Z(t-n) F_Z(s-m) - F_Z(s-n) F_Z(t-m)
\end{cases}
$$

Note that $a_m(m) = b_m(m) = 0$ and the series $\sum_{n=m}^{\infty} a_n(m)$ is convergent. In addition, $b_n(m) \in [-1, 1]$, for all $m, n \in \mathbb{Z}$. Then, by applying Lemma 1, we find

$$
\sum_{n=m+1}^{\infty} a_n(m) b_n(m) = \sum_{n=m}^{\infty} a_n(m) b_n(m) = \sum_{n=m}^{\infty} \left[ b_{n+1}(m) - b_n(m) \right] \sum_{i=n+1}^{\infty} a_i(m) \right].
$$

From (1), we obtain

$$
b_{n+1}(m) - b_n(m) = f_Z(t-n-1) F_Z(s-m) - f_Z(s-n-1) F_Z(t-m) \geq 0, \text{ for all } n \geq m.
$$

On the other hand, by using the characterization of the MRL order given in (4), we get

$$
\sum_{i=n+1}^{\infty} a_i(m) = F_Y(m) \sum_{i=n+1}^{\infty} F_X(i) - F_X(m) \sum_{i=n+1}^{\infty} F_Y(i) \geq 0.
$$

Therefore, $X + Z \succeq_{mrl} Y + Z$. \hfill \square

Lemma 2 leads to a discrete version of Theorem 2.A.9 in [6].

**Theorem 1.** Let $(X_i, Y_i)$, $i = 1, \cdots, n$, be independent pairs of discrete random variables, with finite means, taking on values in $\mathbb{N}$. Suppose that $X_i \succeq_{mrl} Y_i$, $i = 1, \cdots, n$, and $X_i, Y_i$ are all IFR. Then

$$
\sum_{i=1}^{n} X_i \succeq_{mrl} \sum_{i=1}^{n} Y_i.
$$
3 Comparison in the mean residual life order of convolutions of heterogeneous geometric distributions

In this section, we discuss ordering properties of convolutions of geometric distributions. A geometric random variable $X$ with parameter $p \in (0,1)$ has its probability mass function as

$$f_X(k) = P(X = k) = p(1-p)^k, \ k \in \mathbb{N},$$

survival function as

$$F_X(k) = P(X \geq k) = (1-p)^k, \ k \in \mathbb{N},$$

and mean residual life as

$$m_X(k) = \sum_{j=k+1}^{\infty} \frac{F_X(j)}{F_X(k)} = \frac{1-p}{p}, \ k \in \mathbb{N}.$$

Consequently, if $X$ and $Y$ are geometric random variables with respective parameters $p$ and $q$, then

$$X \geq_{mrl} Y \iff p \leq q. \quad (5)$$

The geometric distribution possesses the memoryless property. Moreover, a geometric random variable $X$ with parameter $p$ has a constant hazard rate function, viz., $r_X = p$, and so $X$ is IFR.

Lemma 3. [3] Let $X_1$ and $X_2$ be two independent geometric random variables with respective parameters $p_1$ and $p_2$. Then, the mean residual life of $X_1 + X_2$ is given by

$$m_{X_1+X_2}(k) = \begin{cases} 
\frac{p_2^2(1-p_1)^{k+2}-p_1^2(1-p_2)^{k+2}}{p_1 p_2 (p_1(1-p_1)^{k+1} - p_2(1-p_2)^{k+1})}, & \text{if } p_1 \neq p_2 \\
\frac{(1-p_1)(2+k p_1)}{p_1(1+k p_1)}, & \text{if } p_1 = p_2 
\end{cases}, \text{ for } k \in \mathbb{N}. \quad (6)$$

We need also need the following lemmas for establishing the main results of this section.

Lemma 4. Suppose $k, s, i \in \mathbb{N}$ such that $s \leq k$ and $i \leq s - 1$. Then:

$$\binom{k}{i} \binom{k}{s-i} \leq \binom{k}{i+1} \binom{k}{s-i-1}, \text{ if } i \leq \frac{s-1}{2} \quad (7)$$

$$\binom{k}{i} \binom{k}{s-i} \geq \binom{k}{i+1} \binom{k}{s-i-1}, \text{ if } i \geq \frac{s-1}{2}. \quad (8)$$

Proof. The desired inequalities follow readily from the relation

$$\binom{k}{i+1} \binom{k}{s-i-1} - \binom{k}{i} \binom{k}{s-i} = \frac{(k+1)(s-1-2i)}{(i+1)(k+i+1-s)}.$$

\[ \square \]
Lemma 5. Suppose $k, s, j \in \mathbb{N}, \ k > 0$. Let us denote

$$A_{k,s,j} \equiv (k+1-j) \sum_{i=0}^{j} \binom{k}{i} \binom{k}{s-i} - (k+1-s+j) \sum_{i=0}^{s-j} \binom{k}{i} \binom{k}{s-i}$$

and

$$B_{k,s,j} \equiv (k+1-j) \sum_{i=s-k}^{j} \binom{k}{i} \binom{k}{s-i} - (k+1-s+j) \sum_{i=s-k}^{s-j} \binom{k}{i} \binom{k}{s-i}.$$ 

Then, the following inequalities hold:

1. $A_{k,s,j} \geq 0$ for $s \leq k$ and $s/2 < j \leq s$;
2. $B_{k,s,j} \geq 0$ for $k+1 \leq s \leq 2k$ and $s/2 < j \leq k$.

Proof. (1) Assume $s \leq k$ and $s/2 < j \leq s$. Then, we have

$$A_{k,s,j} = (k+1-j) \sum_{i=s-j+1}^{j} \binom{k}{i} \binom{k}{s-i} - (2j-s) \sum_{i=0}^{s-j} \binom{k}{i} \binom{k}{s-i} \geq (s+1-j)(2j-s) \left[ \frac{\sum_{i=s-j+1}^{j} \binom{k}{i} \binom{k}{s-i}}{2j-s} - \frac{\sum_{i=0}^{s-j} \binom{k}{i} \binom{k}{s-i}}{s+1-j} \right].$$

From the assumption, we get $s-j \leq (s-1)/2 \leq j-1$. Then, by using the inequalities in (7) and (8) of Lemma 4 we obtain

$$\sum_{i=s-j+1}^{j} \binom{k}{i} \binom{k}{s-i} \geq \binom{k}{s-j} \binom{k}{j} \geq \sum_{i=0}^{s-j} \binom{k}{i} \binom{k}{s-i},$$

and so $A_{k,s,j} \geq 0$.

(2) Assume $k+1 \leq s \leq 2k$ and $s/2 < j \leq k$. Then, we have

$$B_{k,s,j} = (k+1-j)(2j-s) \left[ \frac{\sum_{i=s-j+1}^{j} \binom{k}{i} \binom{k}{s-i}}{2j-s} - \frac{\sum_{i=s-k}^{s-j} \binom{k}{i} \binom{k}{s-i}}{k+1-j} \right].$$

Since $s-k \leq s-j \leq (s-1)/2 < j$, Lemma 4 implies

$$\sum_{i=s-j+1}^{j} \binom{k}{i} \binom{k}{s-i} \geq \binom{k}{s-j} \binom{k}{j} \geq \sum_{i=s-k}^{s-j} \binom{k}{i} \binom{k}{s-i},$$

and so $B_{k,s,j} \geq 0$. \qed
Lemma 6. For \( k \in \mathbb{N} \), let the function \( \psi_k : (0, \infty)^2 \to \mathbb{R} \) be defined as
\[
\psi_k(x, y) = \frac{x^{k+2}(1+y)^k - y^{k+2}(1+x)^k}{x^{k+1}(1+y)^k - y^{k+1}(1+x)^k}
\]
for \( x, y > 0 \), \( x \neq y \) and
\[
\psi_k(x, x) = \frac{x(2x + k + 2)}{x + k + 1}
\]
for \( x > 0 \).

Then, \( \psi_k \) is a Schur-convex function for all \( k = 0, 1, \ldots \).

Proof. The result is evident for \( k = 0 \). So, let us suppose \( k > 0 \). From [4], it suffices to show that \( \psi_k \) is a continuous differentiable function with the following property:
\[
(x - y) \left( \frac{\partial \psi_k}{\partial x}(x, y) - \frac{\partial \psi_k}{\partial y}(x, y) \right) \geq 0 \quad \forall \ x, y > 0.
\] (9)

To obtain (9), we present a unitary expression of \( \psi_k \). For \( x, y \in (0, \infty) \), we have \( x^{k+2}(1+y)^k - y^{k+2}(1+x)^k = (x-y)P_k(x, y) \) and \( x^{k+1}(1+y)^k - y^{k+1}(1+x)^k = (x-y)Q_k(x, y) \), where
\[
P_k(x, y) = \sum_{i=0}^{k} \left( \begin{array}{c} k \\ i \end{array} \right) y^i \sum_{j=i}^{k+1} x^j y^{k+1-j}
\]
and
\[
Q_k(x, y) = \sum_{i=0}^{k} \left( \begin{array}{c} k \\ i \end{array} \right) y^i \sum_{j=i}^{k} x^j y^{k-j}.
\]

Since \( P_k(x, y) = yQ_k(x, y) + x^{k+1}(1+y)^k \), we obtain
\[
\psi_k(x, y) = \frac{P_k(x, y)}{Q_k(x, y)} = y + x^{k+1}(1+y)^k \frac{Q_k(x, y)}{Q_k(x, y)} \quad \forall \ (x, y) \in (0, \infty)^2, \ x \neq y.
\] (10)

But, from the definition of \( \psi_k \), we easily note that (10) holds for \( x = y \) as well. Then, \( \psi_k \) is a continuous differentiable function on \( (0, \infty)^2 \) and, for all \( x, y > 0 \), we have
\[
\frac{\partial \psi_k}{\partial x}(x, y) = \frac{\Delta_k(x, y)}{Q_k^2(x, y)},
\]
where
\[
\Delta_k(x, y) \equiv x^{k}(1+y)^k \sum_{i=0}^{k} \left( \begin{array}{c} k \\ i \end{array} \right) y^i \sum_{j=i}^{k} (k+1-j)x^j y^{k-j}
\]
\[
= \sum_{s=0}^{k} \left\{ \sum_{j=0}^{s} (k+1-j)x^{k+j} y^{k+s-j} \sum_{i=0}^{j} \left( \begin{array}{c} k \\ i \end{array} \right) \left( \begin{array}{c} k \\ s-i \end{array} \right) 
\right.
\]
\[
\left. + \sum_{j=s+1}^{k} (k+1-j)x^{k+j} y^{k+s-j} \left( \begin{array}{c} 2k \\ s \end{array} \right) \right\}
\]
\[
+ \sum_{s=k+1}^{2k} \left\{ \sum_{j=s-k}^{k} (k+1-j)x^{k+j} y^{k+s-j} \sum_{i=s-k}^{j} \left( \begin{array}{c} k \\ i \end{array} \right) \left( \begin{array}{c} k \\ s-i \end{array} \right) \right\}.
\]
For \( n, m \in \mathbb{N} \), let us consider the polynomial \( L_{m,n}(x, y) \equiv (xy)^m (x^n - y^n) \). Clearly,
\[
(x - y)L_{m,n}(x, y) \geq 0 \quad \forall \ x, y > 0.
\]
We then have
\[
Q_k^2(x, y) \left( \frac{\partial \psi_k}{\partial x}(x, y) - \frac{\partial \psi_k}{\partial y}(x, y) \right) = \Delta_k(x, y) - \Delta_k(y, x)
\]
\[
= \sum_{s=0}^{k} \left\{ \sum_{j=\lceil s/2 \rceil + 1}^{s} A_{k,s,j} L_{k+s-j,2j-s}(x, y) + \sum_{j=s+1}^{k} (k + 1 - j) \left( \frac{2k}{s} \right) L_{k+s-j,2j-s}(x, y) \right\}
\]
\[
+ \sum_{s=k+1}^{2k} \sum_{j=\lceil s/2 \rceil + 1}^{k} B_{k,s,j} L_{k+s-j,2j-s}(x, y).
\]
Now, by using Lemma 5, we obtain (9) and therefore \( \psi_k \) is a Schur-convex function for all \( k \in \mathbb{N} \).

The following theorem characterizes the mean residual life order of sums of independent geometric random variables by using the reciprocal majorization.

**Theorem 2.** [3]
(a) Let \( (X_1, X_2) \) and \( (Y_1, Y_2) \) be two sets of independent heterogeneous geometric random variables with parameters \((p_1, p_2)\) and \((q_1, q_2)\), respectively. Then
\[
X_1 + X_2 \geq_{mrl} Y_1 + Y_2 \iff (p_1, p_2) \geq_{rm} (q_1, q_2).
\]
(b) Let \( X_1, \ldots, X_n \) and \( Y_1, \ldots, Y_n \) be two sets of independent heterogeneous geometric random variables with parameters \( p_1, \ldots, p_n \) and \( q_1, \ldots, q_n \), respectively.

(i) If \( (p_1, \ldots, p_n) \geq_{rm} (q_1, \ldots, q_n) \), then \( \sum_{i=1}^{n} X_i \geq_{mrl} \sum_{i=1}^{n} Y_i \);

(ii) If \( \sum_{i=1}^{n} X_i \geq_{mrl} \sum_{i=1}^{n} Y_i \), then \( p_{(1)} \leq q_{(1)} \) and \( \sum_{i=1}^{n} p_i^{-1} \geq \sum_{i=1}^{n} q_i^{-1} \).

Our purpose is to present an alternative proof of this theorem.

**Proof of (a)**
Let us assume \( (p_1, p_2) \geq_{rm} (q_1, q_2) \), where \( p_1, p_2, q_1, q_2 \in (0, 1) \). Without loss of generality, suppose that \( p_1 \leq p_2 \) and \( q_1 \leq q_2 \) (i.e., \( p_i = p_{(i)} \) and \( q_i = q_{(i)} \) for \( i = 1, 2 \)). Then,
\[
p_1 \leq q_1 \quad \text{and} \quad \frac{1}{p_1} + \frac{1}{p_2} \geq \frac{1}{q_1} + \frac{1}{q_2}.
\]
Let \( m = \frac{2p_1 p_2}{p_1 + p_2} \in (0, 1) \) denote the harmonic mean of \( p_1 \) and \( p_2 \). We discuss two cases.

**Case 1.** Assume that \( m \leq q_1 \). Note in this case that \( (x_1, y_1) \geq m (x_2, y_2) \), where
\[
x_1 = \frac{1}{p_1} - 1, \quad y_1 = \frac{1}{p_2} - 1 \quad \text{and} \quad x_2 = y_2 = \frac{1}{m} - 1.
\]
are positive numbers. Then, from Lemma 6, we have
\[
\psi_k(x_1, y_1) \geq \psi_k(x_2, y_2) \quad \forall k \in \mathbb{N}.
\]  

Let \( Z_1 \) and \( Z_2 \) be two independent geometric random variables with the common parameter \( m \). In addition, we can suppose that \( (Y_i, Z_i), \ i = 1, 2, \) are independent pairs of random variables. Then, Lemma 3 and Eq. (12) yields \( m_{X_1 + X_2}(k) \geq m_{Z_1 + Z_2}(k) \quad \forall k \in \mathbb{N}, \) and so \( X_1 + X_2 \geq_{\text{mrl}} Z_1 + Z_2. \) Since \( m \leq q_1 \leq q_2, \) we have \( Z_i \geq_{\text{mrl}} Y_i, \ i = 1, 2. \) From Theorem 1, we obtain \( Z_1 + Z_2 \geq_{\text{mrl}} Y_1 + Y_2, \) and hence \( X_1 + X_2 \geq_{\text{mrl}} Y_1 + Y_2. \)

Case 2. Assume that \( m > q_1. \) Denote
\[
r_2 = \frac{1}{\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{q_1}}.
\]

We then have \( 0 < p_1 \leq q_1 < m < r_2 \leq p_2 \leq q_2 \) and \( (p_1^{-1} - 1, p_2^{-1} - 1) \geq (q_1^{-1} - 1, r_2^{-1} - 1). \) Let \( Z \) be a geometric random variable with parameter \( r_2, \) such that \( (Z, Y_i), \ i = 1, 2, \) are independent pairs of random variables. From Lemmas 3 and 6, we find \( X_1 + X_2 \geq_{\text{mrl}} Y_1 + Y_2. \) Since \( Z \geq_{\text{mrl}} Y_2, \) using the above mentioned closure property, we obtain \( Y_1 + Z \geq_{\text{mrl}} Y_1 + Y_2, \) and hence \( X_1 + X_2 \geq_{\text{mrl}} Y_1 + Y_2. \)

Conversely, assume \( X_1 + X_2 \geq_{\text{mrl}} Y_1 + Y_2, \) i.e., \( m_{X_1 + X_2}(k) \geq m_{Y_1 + Y_2}(k) \quad \forall k \in \mathbb{N}. \) Then, for \( k = 0, \) we obtain \( p_1^{-1} + p_2^{-1} - 2 \geq q_1^{-1} + q_2^{-1} - 2. \) We can suppose that \( p_1 \leq p_2 \) and \( q_1 \leq q_2. \) In this case, following Lemma 3, we get
\[
\frac{1}{p_1} - 1 = \lim_{k \to \infty} m_{X_1 + X_2}(k) \geq \lim_{k \to \infty} m_{Y_1 + Y_2}(k) = \frac{1}{q_1} - 1.
\]

So, we get \( (p_1, p_2) \geq_{\text{mrl}} (q_1, q_2). \)

**Proof of (b)**

i) The proof is obtained by induction. The case \( n = 2 \) has already been treated in (a). Let us now suppose that the implication is true for \( n - 1, \) where \( n \geq 3. \) Let us consider \( (p_1, \ldots, p_n) \geq_{\text{mrl}} (q_1, \ldots, q_n), \) where \( p_1, \ldots, p_n \) and \( q_1, \ldots, q_n \) are the parameters of independent geometric random variables \( X_1, \ldots, X_n \) and \( Y_1, \ldots, Y_n, \) respectively. Without loss of generality, we can assume \( p_i \leq p_{i+1} \) and \( q_i \leq q_{i+1} \) for \( i = 1, \ldots, n - 1. \) Then, we have \( \sum_{i=1}^{k} \frac{1}{p_i} \geq \sum_{i=1}^{k} \frac{1}{q_i} \) for \( k = 1, \ldots, n. \)

If \( p_n \leq q_1, \) then \( p_i \leq q_i \) for all \( i. \) Then, from Theorem 1, we obtain \( \sum_{i=1}^{n} X_i \geq_{\text{mrl}} \sum_{i=1}^{n} Y_i. \)

Suppose now \( q_1 < p_n. \) Since \( p_1 \leq q_1, \) there exists a unique \( k \in \{1, \ldots, n - 1\} \) such that \( p_k \leq q_1 < p_{k+1}, \) and consequently \( \frac{1}{p_{k+1}} < \frac{1}{p_k}. \) Let us define a positive number \( \alpha \) by
\[
\alpha = \frac{1}{\frac{1}{p_k} + \frac{1}{p_{k+1}} - \frac{1}{q_1}}.
\]

We have \( \alpha \in (p_k, p_{k+1}] \subset (0, 1). \) Then, it is easy to observe the following orders:
\[
(p_k, p_{k+1}) \geq_{\text{mrl}} (q_1, \alpha) \quad \text{and} \quad (p_1, \ldots, p_{k-1}, \alpha, p_{k+2}, \ldots, p_n) \geq_{\text{mrl}} (q_2, \ldots, q_k, q_{k+1}, \ldots, q_n).
\]
Let $Z$ be a geometric random variable with parameter $\alpha$, which is independent of $X_i$ and $Y_i$, for $i = 1, \cdots, n$. From the induction assumption, it then follows that $X_k + X_{k+1} \geq_{mrl} Y_1 + Z$ and
\[X_1 + \cdots + X_{k-1} + Z + X_{k+1} + \cdots + X_n \geq_{mrl} Y_2 + \cdots + Y_k + Y_{k+1} + \cdots + Y_n.\]

Since convolutions of independent geometric random variables are IFR, we may apply Theorem 1 to obtain the following inequalities:
\[
\sum_{i=1}^{n} X_i \geq_{mrl} \sum_{1 \leq i \leq n; i \neq k, k+1} X_i + Y_1 + Z \geq_{mrl} \sum_{i=1}^{n} Y_i.
\]

Therefore, $\sum_{i=1}^{n} X_i \geq_{mrl} \sum_{i=1}^{n} Y_i$.

(ii) Suppose $\sum_{i=1}^{n} X_i \geq_{mrl} \sum_{i=1}^{n} Y_i$, i.e., $m_{\sum_{i=1}^{n} X_i}(k) \geq m_{\sum_{i=1}^{n} Y_i}(k) \ \forall \ k \in \mathbb{N}$. For $k = 0$, we obtain
\[
\sum_{i=1}^{n} \frac{1-p_i}{p_i} = \sum_{i=1}^{n} E(X_i) = E \left( \sum_{i=1}^{n} X_i \right) = m_{\sum_{i=1}^{n} X_i}(0) \geq m_{\sum_{i=1}^{n} Y_i}(0) = \sum_{i=1}^{n} E(Y_i) = \sum_{i=1}^{n} \frac{1-q_i}{q_i},
\]
and so $\sum_{i=1}^{n} p_i^{-1} \geq \sum_{i=1}^{n} q_i^{-1}$. Assume that $p_i$ ($i = 1, \cdots, n$) are distinct numbers. Using the definition of the mean residual life and a result of [5], we then find
\[
m_{\sum_{i=1}^{n} X_i}(k) = \frac{\sum_{i=1}^{n} p_i^{1-k} (1-p_i)^{n+k} \prod_{1 \leq j \leq n, j \neq i} p_j}{\sum_{i=1}^{n} (1-p_i)^{n+k-1} \prod_{1 \leq j \leq n, j \neq i} p_j p_j - p_i}.
\]

We then easily obtain
\[
\lim_{k \to \infty} m_{\sum_{i=1}^{n} X_i}(k) = \frac{1}{p_1(1)} - 1.
\]

For arbitrary parameters $p_1, p_2, \cdots, p_n$, the same result is obtained by using a limiting argument. Then, $\frac{1}{p_1(1)} = 1 + \lim_{k \to \infty} m_{\sum_{i=1}^{n} X_i}(k) \geq 1 + \lim_{k \to \infty} m_{\sum_{i=1}^{n} Y_i}(k) = \frac{1}{q_1(1)}. \quad \Box$

Let us denote by $H(x_1, \cdots, x_k)$ the harmonic mean of strict positive numbers $x_1, \cdots, x_k$:
\[
H(x_1, \cdots, H_k) = \frac{k}{\frac{1}{x_1} + \cdots + \frac{1}{x_k}}.
\]

We now establish, in terms of the harmonic mean, necessary and sufficient conditions for the comparison in MRL order of the sums of two independent heterogeneous geometric random variables.
Theorem 3. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two sets of independent heterogeneous geometric random variables with the vectors of the associated parameters as $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$, respectively. Let $X_{(i)}$ have the parameter $p_{(i)}$ and $Y_{(i)}$ have the parameter $q_{(i)}$, for $i = 1, \ldots, n$. Then

$$\sum_{i=1}^{k} X_{(i)} \geq_{\text{mrl}} \sum_{i=1}^{k} Y_{(i)} \, (1 \leq k \leq n) \iff H(p_{(1)}, \ldots, p_{(k)}) \leq H(q_{(1)}, \ldots, q_{(k)}) \, (1 \leq k \leq n).$$

Proof. Suppose that $H(p_{(1)}, \ldots, p_{(k)}) \leq H(q_{(1)}, \ldots, q_{(k)})$, for $k = 1, \ldots, n$. Clearly, we have $(p_{(1)}, \ldots, p_{(k)}) \succeq_r (q_{(1)}, \ldots, q_{(k)})$, for all $k \in \{1, \ldots, n\}$. Then, from Theorem 2(b), item (i), we obtain $\sum_{i=1}^{k} X_{(i)} \geq_{\text{mrl}} \sum_{i=1}^{k} Y_{(i)}$, for $1 \leq k \leq n$.

Conversely, assume that $\sum_{i=1}^{k} X_{(i)} \geq_{\text{mrl}} \sum_{i=1}^{k} Y_{(i)}$, for $1 \leq k \leq n$. In this case, from Theorem 2(b), item (ii), we get the following inequalities

$$p_{(1)} \leq q_{(1)} \text{ and } \sum_{i=1}^{k} \frac{1}{p_{(i)}} \geq \sum_{i=1}^{k} \frac{1}{q_{(i)}}, \text{ for } 2 \leq k \leq n.$$

So $H(p_{(1)}, \ldots, p_{(k)}) \leq H(q_{(1)}, \ldots, q_{(k)})$, for $k = 1, \ldots, n$. \hfill \Box

References


